

Quantum contextuality in N -boson systems

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Quantum contextuality in systems of identical bosonic particles is explicitly exhibited via the maximum violation of a suitable inequality of Clauser-Horne-Shimony-Holt type. Unlike the approaches considered so far, which make use of single-particle observables, our analysis involves collective observables constructed using multi-boson operators. An exemplifying scheme to test this violation with a quantum optical setup is also discussed.

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A physical theory is called noncontextual if the measurement results about a given observable A do not depend on whether other commuting observables B are simultaneously measured. Quantum Mechanics turns out to be a contextual theory, Bell's nonlocality being a particular form of contextuality [1–3].

Many experimental tests of contextuality have been proposed, where the commuting observables A and B refer to the spatial and polarization degrees of freedom of single spin- $\frac{1}{2}$ particles or photons [4–14]. Most of the tests check whether inequalities of Clauser-Horne-Shimony-Holt (CHSH) type [15], applied now to local, single-system settings:

$$\mathcal{E} = \left| \langle A(B + B') \rangle + \langle A'(B - B') \rangle \right| \leq 2, \quad (1)$$

are violated or not, $\langle \dots \rangle$ being the expectation values of the involved quantities in a suitable system state. These inequalities are a straightforward consequence of assuming that the values assigned to products AB , AB' , $A'B$ and $A'B'$ of observables A and A' , pertaining to a given degree of freedom, commuting with observables B and B' pertaining to a different degree of freedom, are products of the values independently assigned to the separate observables A , A' , B and B' .

Note that different approaches to investigate quantum contextuality, not based on the inequality (1), have also been considered, for instance, see [16–23]. Furthermore, violation of noncontextuality has been exploited as a resource for quantum key distribution and quantum information processing purposes [24–26]. The contextuality of mesoscopic systems has been recently addressed in systems consisting of distinguishable particles [27] and superconductive devices [28], but all involving single-qubit measurements.

The question which naturally arises is whether it is possible to test noncontextuality in systems of N identical particles, like trapped ultracold bosons, phonons in systems of micro-mechanical oscillators and multiphoton states in quantum optics, *i.e.* in truly mesoscopic systems. In such cases, the use of collective observables is mandatory; from the theoretical point of view, these ob-

servables can be naturally described in terms of multiboson algebras, which represent a novel tool in this specific context. This analysis of mesoscopic systems represents also an important contribution to the study of the transition between quantum and classical systems.

Motivated by the previous considerations, here we address quantum contextuality in systems made of N identical particles (bosons). Our main result is the identification of a set of collective observables for multi-particle tests of noncontextuality. More specifically, using maximally entangled NOON-like states, we first show that inequalities of CHSH type as in (1) can not be violated if constructed with standard single-boson observables; this is ultimately due to the indistinguishability of the involved particles. In order to make quantum contextuality apparent in such systems, one has to use instead collective observables built with suitable multi-boson operators; indeed, with such observables one can construct CHSH like inequalities that are maximally violated by the same NOON-like states. A physical implementation based on a quantum optical interferometric scheme able to test these results is also presented and discussed.

Let us consider a system of N bosons with two degrees of freedom, *e.g.* a “spatial” and an “internal” one. For sake of definiteness, we assume these degrees of freedom to take two possible values, labeled by the variables $k = 1, 2$ and $\alpha = -, +$, respectively. As appropriate for boson systems, we adopt a second quantized description and introduce suitable creation $a_k^{(\alpha)\dagger}$ (and annihilation $a_k^{(\alpha)}$) operators for the mode (k, α) , satisfying the usual bosonic algebra: $[a_k^{(\alpha)}, a_l^{(\beta)\dagger}] = \delta_{k,l} \delta_{\alpha,\beta}$. Out of the vacuum $|0\rangle$, they create the (Fock) states of the system

$$|n_1^{(-)}, n_2^{(-)}, n_1^{(+)}, n_2^{(+)}\rangle = \prod_{k=\{1,2\}} \prod_{\alpha=\{-,+\}} \frac{(a_k^{(\alpha)\dagger})^{n_k^{(\alpha)}}}{\sqrt{n_k^{(\alpha)}!}} |0\rangle, \quad (2)$$

containing $n_k^{(\alpha)}$ bosons in the mode (k, α) , such that $\sum_{k,\alpha} n_k^{(\alpha)} = N$. For sake of compactness, hereafter we shall indicate only the nonvanishing occupation numbers, *i.e.*, those for which $n_k^{(\alpha)} \neq 0$.

A basic example of a quantum system exhibiting contextuality, that is violating the expectation that the result of a measurement be independent of a previous or simultaneous measurement of any set of mutually commuting observables, is a single particle system ($N = 1$), prepared in the following superposition of Fock states:

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}} \left(a_1^{(-)\dagger} + a_2^{(+)\dagger} \right) |0\rangle \\ &= \frac{1}{\sqrt{2}} \left(|n_1^{(-)} = 1\rangle + |n_2^{(+)} = 1\rangle \right). \end{aligned} \quad (3)$$

By taking the variables k and α to label suitable spin-like components, one finds that CHSH-like inequalities as in (1) are indeed maximally violated, as confirmed in experiments based on single-neutron [6] and single-photon interferometers [7, 8].

Similarly, in the case of N bosons one can consider the following spin-like observables

$$J_x = \frac{1}{2} \sum_{\alpha=\{-,+\}} \left[a_1^{(\alpha)\dagger} a_2^{(\alpha)} + a_2^{(\alpha)\dagger} a_1^{(\alpha)} \right], \quad (4a)$$

$$J_y = \frac{i}{2} \sum_{\alpha=\{-,+\}} \left[a_2^{(\alpha)\dagger} a_1^{(\alpha)} - a_2^{(\alpha)\dagger} a_1^{(\alpha)} \right], \quad (4b)$$

$$J_z = \frac{1}{2} \sum_{\alpha=\{-,+\}} \left[a_1^{(\alpha)\dagger} a_1^{(\alpha)} - a_2^{(\alpha)\dagger} a_2^{(\alpha)} \right], \quad (4c)$$

$$S_x = \frac{1}{2} \sum_{k=\{1,2\}} \left[a_k^{(-)\dagger} a_k^{(+)} + a_k^{(+)\dagger} a_k^{(-)} \right], \quad (4d)$$

$$S_y = \frac{i}{2} \sum_{k=\{1,2\}} \left[a_k^{(+)\dagger} a_k^{(-)} - a_k^{(-)\dagger} a_k^{(+)} \right], \quad (4e)$$

$$S_z = \frac{1}{2} \sum_{k=\{1,2\}} \left[a_k^{(-)\dagger} a_k^{(-)} - a_k^{(+)\dagger} a_k^{(+)} \right]. \quad (4f)$$

The observables J_i and S_i both satisfy the $SU(2)$ algebraic relations ($[J_i, J_j] = i\epsilon_{ijk} J_k$, and similarly for S_i), while commuting among themselves, $[J_i, S_j] = 0$. Furthermore, their eigenvalues are the integers between $-N/2$ and $N/2$. Then, once rescaled by $2/N$, their eigenvalues lie in the interval $[-1, 1]$ and the rescaled observables should violate a suitable constructed inequality of CHSH type of the form (1).

In the case of the single-boson system ($N = 1$), this can be easily shown by taking the following choices for the four observables appearing in the inequality (1):

$$\begin{aligned} A &= 2 J_z, & B &= 2 S_z(\pi/4), \\ A' &= 2 J_x, & B' &= 2 S_z(-\pi/4), \end{aligned}$$

where $S_z(\theta) = \cos \theta S_z + \sin \theta S_x$. For the state (3), straightforward calculations lead to:

$$\langle J_z \otimes S_z \rangle = \langle J_x \otimes S_x \rangle = 1/4, \quad (5)$$

$$\langle J_z \otimes S_x \rangle = \langle J_x \otimes S_z \rangle = 0, \quad (6)$$

whence $\mathcal{E} = 2\sqrt{2} > 2$ indeed exhibits the maximum violation of the CHSH inequality allowed by quantum mechanics [29].

For systems of bosons with $N > 1$, the obvious generalization of these single-particle choices does not lead to any noncontextuality violation; this is not surprising, since in order to exhibit quantum contextuality the CHSH test, $\mathcal{E} > 2$, needs to be adapted to the system under study through a careful choice of both the state and of the four observables A, A', B and B' . In the case of N bosons, the state (3) can be easily generalized:

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2N!}} \left[(a_1^{(-)\dagger})^N + (a_2^{(+)\dagger})^N \right] |0\rangle \\ &= \frac{1}{\sqrt{2}} \left[|n_1^{(-)} = N\rangle + |n_2^{(+)} = N\rangle \right]; \end{aligned} \quad (7)$$

this is a NOON-like state, superposition of two Fock states, the first representing N bosons all in the same mode with $k = 1, \alpha = -$ and similarly the other, N bosons in the state with $k = 2, \alpha = +$; it clearly reduces to the state (3) when $N = 1$. However, the averages of the same observables as before in this new state $|\Psi\rangle$ become:

$$\langle J_z \otimes S_z \rangle = N^2/4, \quad (8)$$

$$\langle J_x \otimes S_x \rangle = \langle J_z \otimes S_x \rangle = \langle J_x \otimes S_z \rangle = 0; \quad (9)$$

as a consequence, after the necessary rescaling of the spin-like observables by $2/N$, one finds $\mathcal{E} = \sqrt{2} < 2$ and therefore no violation of the noncontextuality test.

Consider instead the multi-boson operators [30–32]:

$$A_k^{(\alpha)} = F_N(\hat{n}_k^{(\alpha)}) (a_k^{(\alpha)})^N, \quad (10)$$

where

$$F_N(\hat{n}_k^{(\alpha)}) = \left[\left[\frac{\hat{n}_k^{(\alpha)} + N}{N} \right] \frac{\hat{n}_k^{(\alpha)}!}{(\hat{n}_k^{(\alpha)} + N)!} \right]^{1/2}, \quad (11)$$

with $\hat{n}_k^{(\alpha)} = a_k^{(\alpha)\dagger} a_k^{(\alpha)}$ the number operator relative to the mode (k, α) , the symbol $\llbracket \cdot \rrbracket$ representing the integer part. The $A_k^{(\alpha)}$ are collective annihilation operators as they diminish by N the number of bosons; indeed, on the orthonormal basis of Fock number states as in (2), they act as follows:

$$A_k^{(\alpha)} |n_k^{(\alpha)}\rangle = \sqrt{\left\llbracket \frac{n_k^{(\alpha)}}{N} \right\rrbracket} |n_k^{(\alpha)} - N\rangle, \quad (12)$$

while the Hermitian conjugate operators $A_k^{(\alpha)\dagger}$ act as collective creation operators:

$$A_k^{(\alpha)\dagger} |n_k^{(\alpha)}\rangle = \sqrt{\left\llbracket \frac{n_k^{(\alpha)}}{N} \right\rrbracket + 1} |n_k^{(\alpha)} + N\rangle. \quad (13)$$

Then, one easily finds that: $[A_k^{(\alpha)}, A_l^{(\beta)\dagger}] = \delta_{k,l} \delta_{\alpha,\beta}$.

In analogy with (4), with these multi-boson annihilation and creation operators one can now form collective spin-like operators relative to the spatial degree of freedom:

$$\mathcal{J}_x = \frac{1}{2} \sum_{\alpha=\{-,+\}} \left[A_1^{(\alpha)\dagger} A_2^{(\alpha)} + A_2^{(\alpha)\dagger} A_1^{(\alpha)} \right], \quad (14a)$$

$$\mathcal{J}_y = \frac{i}{2} \sum_{\alpha=\{-,+\}} \left[A_2^{(\alpha)\dagger} A_1^{(\alpha)} - A_2^{(\alpha)\dagger} A_1^{(\alpha)} \right], \quad (14b)$$

$$\mathcal{J}_z = \frac{1}{2} \sum_{\alpha=\{-,+\}} \left[A_1^{(\alpha)\dagger} A_1^{(\alpha)} - A_2^{(\alpha)\dagger} A_2^{(\alpha)} \right]; \quad (14c)$$

one checks that they satisfy the $SU(2)$ commutation relations, $[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk} \mathcal{J}_k$. In the same way, the multi-boson, collective spin-like polarization operators

$$\mathcal{S}_x = \frac{1}{2} \sum_{k=\{1,2\}} \left[A_k^{(-)\dagger} A_k^{(+)} + A_k^{(+)\dagger} A_k^{(-)} \right], \quad (15a)$$

$$\mathcal{S}_y = \frac{i}{2} \sum_{k=\{1,2\}} \left[A_k^{(+)\dagger} A_k^{(-)} - A_k^{(-)\dagger} A_k^{(+)} \right], \quad (15b)$$

$$\mathcal{S}_z = \frac{1}{2} \sum_{k=\{1,2\}} \left[A_k^{(-)\dagger} A_k^{(-)} - A_k^{(+)\dagger} A_k^{(+)} \right]. \quad (15c)$$

satisfy $[\mathcal{S}_i, \mathcal{S}_j] = i\epsilon_{ijk} \mathcal{S}_k$. Further, as before, the two sets of operators mutually commute, $[\mathcal{J}_i, \mathcal{S}_j] = 0$.

Further, notice that

$$A_k^{(\alpha)\dagger} A_k^{(\alpha)} = \left\lfloor \frac{\hat{n}_k^{(\alpha)}}{N} \right\rfloor, \quad (16)$$

so that, unlike the single-boson spin-like operators in (4), the operators in (14) and (15) have eigenvalues $\pm 1/2$. Thus, we can directly consider the inequality (1) relative to the NOON state (7) and to the collective observables

$$A = 2 \mathcal{J}_z, \quad B = 2 \mathcal{S}_z(\pi/4), \quad (17a)$$

$$A' = 2 \mathcal{J}_x, \quad B' = 2 \mathcal{S}_z(-\pi/4), \quad (17b)$$

where $\mathcal{S}_z(\theta) = \cos \theta \mathcal{S}_z + \sin \theta \mathcal{S}_x$. One finds:

$$\langle \mathcal{J}_z \otimes \mathcal{S}_z \rangle = \langle \mathcal{J}_x \otimes \mathcal{S}_x \rangle = 1/4 \quad (18)$$

$$\langle \mathcal{J}_z \otimes \mathcal{S}_x \rangle = \langle \mathcal{J}_x \otimes \mathcal{S}_z \rangle = 0, \quad (19)$$

whence $\mathcal{E} = 2\sqrt{2} > 2$, so that the CHSH inequality (1) results maximally violated and quantum contextuality is manifest.

An example of physical setup able to test, in principle, this violation can be built within quantum optics. In this framework, the two degrees of freedom labeled by k and α can be taken to refer to the photon path (momentum) and polarization: the operator $a_k^{(\alpha)\dagger}$ creates from the vacuum a single photon along the path k with polarization α , while $A_k^{(\alpha)\dagger}$ creates N photons in the same

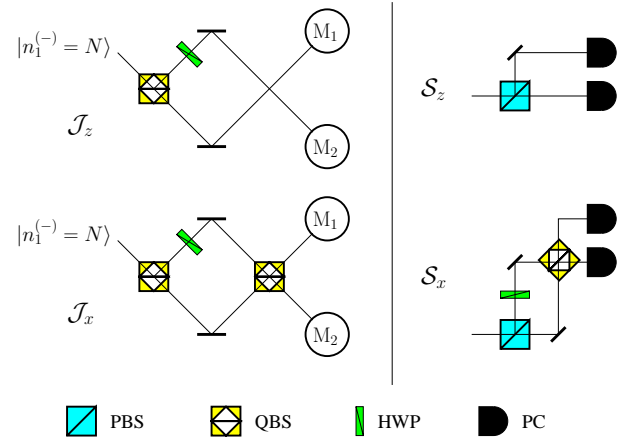


FIG. 1: Physical implementation of a quantum optics setup able to test quantum contextuality with NOON-like states. Legend of the components: PBS = Polarizing Beam Splitter, QBS = Quantum Beam Splitter, HWP = Half-Wave Plate, PC = Photon Counter. See the text for details.

state. Similarly, unlike J_i and S_i in (4), the operators \mathcal{J}_i in (14) and \mathcal{S}_i in (15) are collective observables, referring to path and polarization degrees of freedom, respectively.

Notice that the experimental implementation of the test does not require the actual realization of the multi-boson operators $A_k^{(\alpha)\dagger}$ and $A_k^{(\alpha)}$, but only of suitable procedures for measuring the collective observables \mathcal{J}_i and \mathcal{S}_i . In order to achieve this, operations acting on the N -photon states as a whole are nevertheless needed; as a consequence, linear optical passive devices cannot directly be used.

The proposed implementation, sketched in Fig.1, is based on a Mach-Zehnder-like interferometric scheme; however, the usual beam splitters are replaced with the “quantum beam splitters” (QBSs) proposed in [33], where a nonlinear medium is inserted in one of the two arms. These quantum devices, realized through the unitary operator U_{QBS} , generates NOON-like states by acting on a Fock state where the N photons are all prepared in a given path-polarization configuration (k, α) [33]:

$$U_{\text{QBS}} |n_k^{(\alpha)} = N\rangle = \frac{|n_k^{(\alpha)} = N\rangle + (-1)^l |n_l^{(\alpha)} = N\rangle}{\sqrt{2}}, \quad (20)$$

with $l \neq k$ and $k, l = 1, 2$. Here, $k = 1$ ($k = 2$) refers to the lower (upper) path, while $\alpha = -, +$ to the two possible horizontal and vertical polarization, *i.e.* $H \equiv -, V \equiv +$. The NOON-like state given in (7) can then be obtained from the appropriate state (20) with $k = 1$ and $\alpha = -$, by switching the polarization in the upper ($k = 2$) path by means of a half-wave plate (HWP). [It is worth noticing that, due to the presence of the two mirrors which exchange the modes 1 and 2 (see Fig.1), the lower (upper) path is detected as $k = 2$ ($k = 1$); in order

to avoid confusion, in the Fig. 1 we call the detectors M_k , with k referring to the actual followed path.]

The operator \mathcal{J}_z can then be measured by sending the outgoing beams directly to the two detector devices M_k (upper-left scheme in Fig.1); indeed, the detection operation at M_1 and M_2 , for instance implemented by photon counters, precisely corresponds to the projection onto the eigenstates of \mathcal{J}_z . Instead, in the case of the observable \mathcal{J}_x one needs to make the two beams interfere at another QBS (lower-left scheme in Fig.1) before detection at M_k . Using (20), the outgoing state after the second QBS reads now

$$|\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}} \left(|\Psi_2^{(-)}\rangle - |\Psi_1^{(+)}\rangle \right), \quad (21)$$

where $|\Psi_1^{(+)}\rangle = (|n_1^{(+)} = N\rangle - |n_2^{(+)} = N\rangle)/\sqrt{2}$ and $|\Psi_2^{(-)}\rangle = (|n_1^{(-)} = N\rangle + |n_2^{(-)} = N\rangle)/\sqrt{2}$ are eigenstates of \mathcal{J}_x , the label k in $|\Psi_k^{(\pm)}\rangle$, $k = 1, 2$, referring now to the upper and lower exiting paths, beyond the additional QBS. As a result, the detection operation at M_1 and M_2 corresponds to projections onto these two states.

In order to simultaneously determine, together with \mathcal{J}_z , $i = x, z$, also the operators \mathcal{S}_z and \mathcal{S}_x , as required by the inequality (1), the detection procedures at M_k need to be more structured than those provided by simple photon counters. The measurement of \mathcal{S}_z can be implemented through a polarizing beam splitter (PBS) and two photon counters (PCs), that allows to discriminate between $|n_k^{(-)} = N\rangle$ and $|n_k^{(+)} = N\rangle$, eigenstates of \mathcal{S}_z (see the right-upper scheme in Fig.1). Instead, in order to measure \mathcal{S}_x , one needs to insert before the photon counters a further Mach-Zehnder interferometric circuit, built with additional HWP and QBS (see the right-lower scheme of the Fig. 1). From the same considerations described before, in explaining the determination of \mathcal{J}_x , now applied to the polarization degrees of freedom, one can check that photon detection at the exit arms of this additional interferometer precisely corresponds to projection onto the eigenstates of \mathcal{S}_x , $(|n_k^{(-)} = N\rangle \pm |n_k^{(+)} = N\rangle)/\sqrt{2}$.

Finally, it is worth noticing that the schemes just illustrated for measuring the relevant \mathcal{J}_i and \mathcal{S}_i observables are such that all the N photons arrive at one and only one of the photon counters. Therefore, by suitably composing one of the two detecting schemes in the right panel of the Fig. 1 with one of the measurements of the left panel, one can experimentally determine all collective observables needed to test the CHSH inequality discussed above.

Summarizing, in this Letter we discussed a test of quantum contextuality in systems composed by N identical bosonic particles. We showed that, unlike the analysis so far considered in the literature, based on systems of distinguishable particles using single-particle observables, in order to make quantum contextuality apparent

in such mesoscopic bosonic systems, measurements of collective observables are needed. We further remark that the observables we used are built out of collective operators belonging to a multiboson algebra, a tool that has never been considered in such a context. Incidentally, a state-independent test of noncontextuality with N -boson systems can also be straightforwardly achieved with our collective observables following, *e.g.* Refs. [11, 13].

We are confident that these theoretical results will open new perspectives in the study of quantum contextuality in mesoscopic systems, as trapped ultracold bosons and phonons in systems of micro-mechanical oscillators, and provide new insights in the actual realization of further experimental tests, possibly paving the way to new applications in quantum technology.

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- [1] S. Kochen and E.P. Specker, J. Math. Mech. **17**, 59 (1967).
 - [2] N.D. Mermin, Rev. Mod. Phys. **65**, 803 (1993).
 - [3] M. Genovese, Phys. Rep. **413**, 319 (2005).
 - [4] S. Basu *et al.*, Phys. Lett. A **279**, 281 (2001).
 - [5] F. Benatti *et al.* J. Phys. A: Math. Gen. **35**, 4955 (2002).
 - [6] Y. Hasegawa *et al.*, Nature **425**, 45 (2005); Phys. Rev. Lett. **97**, 230401 (2006).
 - [7] M. Michler, H. Weinfurter and M. Żukowski, Phys. Rev. Lett. **84**, 5457 (2000).
 - [8] B.R. Gadway *et al.* J. Phys. B **42**, 015503 (2009).
 - [9] E. Karimi *et al.*, Phys. Rev. A **82**, 022115 (2010).
 - [10] A.K. Pan and D. Home, Phys. Lett. A **373**, 3430 (2009); Int. J. Theor. Phys. **49**, 1920 (2010).
 - [11] A. Cabello, Phys. Rev. Lett. **101**, 210401 (2008).
 - [12] A. Cabello *et al.*, Phys. Rev. Lett. **100** 130404 (2008).
 - [13] E. Amsellem *et al.*, Phys. Rev. Lett. **103** 160405 (2009).
 - [14] C. Simon *et al.*, Phys. Rev. Lett. **85**, 1783 (2000).
 - [15] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
 - [16] N.D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
 - [17] G. Kirchmair *et al.*, Nature **460**, 494 (2009).
 - [18] H. Bartosik *et al.*, Phys. Rev. Lett. **103**, 040403 (2003).
 - [19] B.H. Lim *et al.*, Phys. Rev. A **80**, 044101 (2009).
 - [20] Y. Huang *et al.*, Phys. Rev. Lett. **90**, 250401 (2003).
 - [21] O. Gühne *et al.*, Phys. Rev. A **81**, 022121 (2010).
 - [22] A.R. Plastino and A. Cabello, Phys. Rev. A **82**, 022114 (2010).
 - [23] G. McKeown *et al.*, arXiv:1102.5297.
 - [24] S. Adhikari *et al.*, arXiv:1101.0660.
 - [25] M. Kleinmann *et al.*, arXiv:1007.3650.
 - [26] K. Horodecki *et al.*, arXiv:1006.0468.
 - [27] A. Cabello, Phys. Rev. A **82**, 032110 (2010).
 - [28] L.F. Wei *et al.*, Phys. Rev. B **81**, 174513 (2010).
 - [29] B.S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).
 - [30] R.A. Brandt and O.W. Greenberg, J. Math. Phys. **10**, 1168 (1969).
 - [31] M. Rasetti, Int. J. Theor. Phys. **5**, 377 (1972).
 - [32] F. Raffa and M. Rasetti, Int. J. Quant. Inf. **7** 221 (2009); M. Genovese, F. A. Raffa, M. Rasetti, arXiv:0912.4582.
 - [33] J. Dunningham and T. Kim, J. Mod. Opt. **53**, 557 (2006).